

# A CLASSIFICATION OF HOMOGENEOUS OPERATORS IN THE COWEN-DOUGLAS CLASS

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**ABSTRACT.** An explicit construction of all the homogeneous holomorphic Hermitian vector bundles over the unit disc  $\mathbb{D}$  is given. It is shown that every such vector bundle is a direct sum of irreducible ones. Among these irreducible homogeneous holomorphic Hermitian vector bundles over  $\mathbb{D}$ , the ones corresponding to operators in the Cowen-Douglas class  $B_n(\mathbb{D})$  are identified. The classification of homogeneous operators in  $B_n(\mathbb{D})$  is completed using an explicit realization of these operators. We also show how the homogeneous operators in  $B_n(\mathbb{D})$  split into similarity classes.

## 1. INTRODUCTION

An operator  $T$  is said to be *homogeneous* if its spectrum is contained in the closed unit disc and for every Möbius transformation  $g$  of the unit disc  $\mathbb{D}$ , the operator  $g(T)$  defined via the usual holomorphic functional calculus, is unitarily equivalent to  $T$ . To every homogeneous irreducible operator  $T$  there corresponds (cf. [1, Theorem 2.2]) an *associated projective unitary representation*  $U$  of the Möbius group  $G_0$ :

$$U_g^* T U_g = g(T), \quad g \in G_0.$$

The projective unitary representations of  $G_0$  lift to unitary representations of the universal cover  $\widetilde{G}_0$  which are quite well-known. We can choose (cf. [1, Lemma 3.1])  $U_g$  such that  $k \mapsto U_k$  is a representation of the rotation group. If

$$\mathcal{H}(n) = \{x \in \mathcal{H} : U_{k_\theta} x = e^{in\theta} x\},$$

where  $k_\theta(z) = e^{i\theta}z$ , then  $T : \mathcal{H}(n) \rightarrow \mathcal{H}(n+1)$  is a block shift. A complete classification of these for  $\dim \mathcal{H}(n) \leq 1$  was obtained in [1] using the representation theory of  $\widetilde{G}_0$ . First examples for  $\dim \mathcal{H}(n) = 2$  appeared in [14]. Recently [7, 9], an  $m$  - parameter family of examples with  $\dim \mathcal{H}(n) = m$  was constructed. We will use the ideas of [7, 9] to obtain a complete classification of the homogeneous operators in the Cowen-Douglas class. Finally, we describe the similarity classes within the homogeneous Cowen-Douglas operators. As a consequence, we obtain an affirmative answer to the Halmos question (cf. [10]) for this class of operators. We also include a somewhat new conceptual presentation of the Cowen-Douglas theory and a brief description of the method of holomorphic induction, which will be our main tool. Our paper is essentially self contained and can be read without the knowledge of [7] and [9]. The results of this paper were announced in [8] except for Theorem 4.2.

**1.1. Vector bundles.** Let  $M$  be a complex manifold and suppose  $\pi : E \rightarrow M$  is a complex vector bundle. We write, as usual,  $E_z = \pi^{-1}(z)$ . For a trivialization,  $\varphi : E \rightarrow M \times \mathbb{C}^n$ , we write  $\varphi(v) = (z, \varphi_z(v))$  for  $v \in E_z$  with  $\varphi_z : E_z \rightarrow \mathbb{C}^n$  linear. (All we are going to say here would be valid using local trivializations, but in this article we will always work with global trivializations.)

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We write  $E_z^*$  for the complex anti-linear dual of  $E_z$ ,  $z \in M$ , and we write  $[u, v]$  for  $u(v)$ ,  $u \in E_z^*$ ,  $v \in E_z$ . We consider  $\mathbb{C}^n$  to be equipped with its natural inner product and identify it with its own anti-linear dual (so  $\xi \in \mathbb{C}^n$  is identified with the anti-linear map  $\eta \mapsto \langle \xi, \eta \rangle_{\mathbb{C}^n}$ ). Then  $\varphi_z^* : \mathbb{C}^n \rightarrow E_z^*$  is well-defined. We set  $\psi_z = \varphi_z^{*-1}$  and  $\psi(u) = (z, \psi_z(u))$  for  $u \in E_z^*$ . This makes  $E^*$  a complex vector bundle with trivialization  $\psi$ . We call  $\varphi$  and  $\psi$ , the *associated trivializations* of  $E$  and  $E^*$ . If  $E$  is a holomorphic vector bundle then  $E^*$  is an anti-holomorphic vector bundle (meaning that for any two trivializations,  $\psi_\alpha$  and  $\psi_\beta$ , the transition functions  $z \mapsto (\psi_\alpha)_z \circ (\psi_\beta)_z^{-1}$  are anti-holomorphic) and vice-versa.

If  $E$  has a Hermitian structure, we automatically equip  $E^*$  with the dual structure (giving the dual norm of  $E_z$  to  $E_z^*$  for all  $z \in M$ ).

By an automorphism of  $\pi : E \rightarrow M$ , we mean a diffeomorphism  $\hat{g} : E \rightarrow E$  such that  $\pi \circ \hat{g} = g \circ \pi$  for some automorphism  $g$  of  $M$ . We write  $g_z$  for the restriction of  $\hat{g}$  to  $E_z$ . The automorphism  $\hat{g}$  also acts on the sections  $f$  of  $E$ , by  $(\hat{g}^* f)(z) = g_z^{-1} f(gz)$ . When  $G$  is the group of automorphisms of  $E$  (acting on the left, as usual) we have a representation  $U$  of  $G$  on the sections given by  $U_{\hat{g}} f = (\hat{g}^{-1})^* f$ , that is,

$$(U_{\hat{g}} f)(z) = g_z f(g^{-1} z).$$

Given an automorphism  $g$  of  $E$ , there is a corresponding automorphism of  $E^*$ , where the place of  $g_z$  is taken by  $g_z^{*-1}$ . This also remains true in the category of Hermitian bundles. It follows that a group  $G$  of automorphisms of  $E$  also acts as a group of automorphisms of  $E^*$ . If  $E$  is *homogeneous*, that is, the action of  $G$  is transitive on  $M$ , then so is  $E^*$ , and vice-versa.

**1.2. Reproducing kernel.** We describe, essentially following [2], how the usual formalism of reproducing kernels can be adapted to vector bundles. Suppose  $\mathcal{H}$  is a Hilbert space whose elements are sections of a vector bundle  $E \rightarrow M$  and suppose the maps  $\text{ev}_z : \mathcal{H} \rightarrow E_z$  are continuous for all  $z \in M$ . Then setting  $K_z = \text{ev}_z^*$ , we have

$$[u, f(z)] = [u, \text{ev}_z(f)] = \langle K_z u, f \rangle_{\mathcal{H}}, \quad u \in E_z, f \in \mathcal{H}. \quad (1.1)$$

For all  $w \in M$  then  $K_w u$  is in  $\mathcal{H}$  and is linear in  $u$ . So, we can write  $K_w(z)u = \text{ev}_z(K_w u) = \text{ev}_z \text{ev}_w^*(u)$ . We also write  $K(z, w) = K_w(z) = \text{ev}_z \text{ev}_w^*$  which is a linear map  $E_w^* \rightarrow E_z$ , and is called the reproducing kernel of  $\mathcal{H}$ , (1.1) is the reproducing property.

Clearly,  $K(w, z) = K(z, w)^*$ . We have the positivity  $\sum_{j,k} [u_k, K(z_k, z_j) u_j] \geq 0$  for any  $z_1, \dots, z_p$  in  $M$  and  $u_1, \dots, u_p \in E_z^*$  which is nothing but the inequality

$$\sum_{j,k} \langle (\text{ev}_{z_k})^* u_k, (\text{ev}_{z_j})^* u_j \rangle_{\mathcal{H}} \geq 0.$$

Conversely, a  $K$  with these properties is always the reproducing kernel of a Hilbert space of sections of  $E$  (cf. [2]).

Suppose we have a vector bundle  $E$  and a Hilbert space  $\mathcal{H}$  of sections of  $E$  with reproducing kernel  $K$ ; suppose  $\hat{g}$  is an automorphism of  $E$ . Then  $\hat{g}$  acts on the sections of  $E$  by  $(\hat{g}^* f)(z) = g_z^{-1} f(gz)$ . By the density of linear combinations of the sections of the form  $K_w u$ , the condition for this action to preserve  $\mathcal{H}$  and act on it isometrically is

$$\langle \hat{g}^*(K_w u), K_z u' \rangle_{\mathcal{H}} = \langle K_w u, (g^{-1})^*(K_z u') \rangle_{\mathcal{H}}$$

for all  $z, w; u, u'$ . Evaluating both sides using (1.1), this amounts to

$$K(gz, gw) = g_z K(z, w) g_w^*, \quad \text{for all } z, w \in M.$$

The following remarks will be important for us. Suppose each  $\text{ev}_z$  is non-singular, that is, its range is the whole of  $E_z$ . (This is so in the important case where  $\mathcal{H}$  is dense in the space of sections of  $E$  in the topology of uniform convergence on compact sets.) Then  $K_z = \text{ev}_z^*$  is an imbedding of  $E_z^*$

into  $\mathcal{H}$ . Postulating that this imbedding is an isometry we obtain a canonical Hermitian structure on  $E^*$ . Using (1.1) we can write for the norm on  $E^*$

$$\|u\|_z^2 = \|K_z u\|_{\mathcal{H}}^2 = [u, K(z, z)u], \quad u \in E_z^*.$$

The vector bundle  $E$  has the dual Hermitian structure, for  $v \in E_z$  we have

$$\|v\|_z^2 = [K(z, z)^{-1}v, v].$$

In fact this statement amounts to

$$|[u, v]|^2 \leq [K(z, z)^{-1}v, v][u, K(z, z)u]$$

for all  $u, v$  with equality reached for some  $u, v$ . Since  $K(z, w)$  is bijective by hypothesis, any  $v \in E_z$  can be written as  $v = K(z, z)u'$  with  $u' \in E_z^*$  and the inequality to be proved is equivalent to  $|[u, K(z, z)u']|^2 \leq [u', K(z, z)u'] [u, K(z, z)u]$ . But this is just the Cauchy-Schwarz inequality.

When  $E$  is a holomorphic vector bundle,  $K(z, w)$  depends in  $z$  holomorphically and on  $w$  anti-holomorphically. Hence  $K(z, w)$  is completely determined by  $K(z, z)$ . It follows that  $K(z, w)$  is completely determined by the canonical Hermitian structure of  $E$  (or  $E^*$ ).

In the last paragraphs, we had a Hilbert space  $\mathcal{H}$  of sections of  $E$  and (under the assumption that each  $\text{ev}_z$  is surjective) we associated to it a family of imbeddings of  $E_z^*$ , the fibres of  $E^*$ , into  $\mathcal{H}$ . This procedure can be reversed which is of importance for what follows. Suppose now that  $E$  is a vector bundle and the fibres  $E_z^*$  of  $E^*$  form a smooth family of subspaces of some Hilbert space  $H$  which together span  $H$ , that is,  $E^*$  is a anti-holomorphic sub-bundle of the trivial bundle  $M \times H$ ). We write  $\iota_z : E_z^* \rightarrow H$  for the (identity) imbeddings. We define,  $\tilde{f}(z) = \iota_z^* f$  for  $f \in H$ ,  $z \in M$ . Then  $\tilde{f}$  is a section of  $E$  and  $\text{ev}_z(\tilde{f}) = \iota_z^* f$ . If we denote by  $\mathcal{H}$  the Hilbert space of all  $\tilde{f}$ ,  $f \in H$ , with norm  $\|\tilde{f}\| = \|f\|$ , each  $\text{ev}_z$  is continuous, so we have a reproducing kernel Hilbert space. The reproducing kernel is  $K_z u = \widetilde{\iota_z u}$ .

**1.3. Operators in the Cowen-Douglas class.** We modify the definition of the class of operators introduced in [4] in an inessential way. A conceptual presentation in which the role of the dual of the bundle constructed in [4] is apparent follows. Given a domain  $\Omega \subseteq \mathbb{C}$ , we say the bounded operator  $T$  on the Hilbert space  $H$  is in  $B_n(\Omega)$  if  $\bar{z}$  is an eigenvalue of  $T$ , the range of the operator  $T - \bar{z}$  is closed, and the corresponding eigenspaces  $F_z$  are of constant dimension  $n$  for  $z \in \Omega$ . It is proved in [4] that the spaces  $F_z$  span an anti-holomorphic Hermitian vector bundle  $F \subseteq \Omega \times H$ . (In [4] the eigenvalues are  $z \in \Omega$  and so  $F$  is a holomorphic vector bundle; it is more convenient for us to change this.) We write, for  $z \in \Omega$ ,  $\iota_z : F_z \rightarrow H$  for the identity imbedding. Now,  $E = F^*$  is a holomorphic vector bundle, this will be the primary object for us. The bundle  $F$  is identified with  $E^*$ , in what follows we refer to it as  $E^*$ . We are now in the situation discussed above in Section 1.2.

To the elements  $f$  of  $H$  there correspond the sections  $\tilde{f}$  of  $E$  (defined by  $\tilde{f}(z) = \iota_z^* f$ ) and form a Hilbert space  $\mathcal{H}$  isomorphic with  $H$  and having a reproducing kernel  $K_z u = \widetilde{\iota_z u}$ .

Under this isomorphism, the operator on  $\mathcal{H}$  corresponding to  $T$  is  $M^*$ , where  $M$  is the multiplication operator  $(M\tilde{f})(z) = z\tilde{f}(z)$ . In fact (cf. [4]) for any  $u \in E_z^*$ ,

$$\begin{aligned} [u, \widetilde{T^* f}(z)] &= \langle \iota_z u, T^* f \rangle_H = \langle T \iota_z u, f \rangle_H = \bar{z} \langle \iota_z u, f \rangle_H \\ &= [u, z\tilde{f}(z)] = [u, M\tilde{f}(z)] \end{aligned}$$

**1.4. Trivialization.** Finally, we describe how the preceding material appears when the vector bundle is trivialized. We always use associated trivializations  $\varphi, \psi$  of  $E$  and  $E^*$ . As explained in the beginning, this means that  $\psi_z = \varphi_z^*{}^{-1}$ , that is,  $[u, v] = \langle \psi_z u, \varphi_z v \rangle_{\mathbb{C}^n}$  for  $u \in E_z^*$  and  $v \in E_z$ . We will consider here only the case where  $E$  is a holomorphic vector bundle. When  $g$  is an automorphism of  $E$ , in the

trivialization  $g_z : E_z \rightarrow E_{gz}$  becomes  $\varphi_{gz} \circ g_z \circ \varphi_z^{-1}$ , which we write as the matrix  $J_g(z)^{-1}$ . When  $g$  is followed by another automorphism  $h$ , the relation  $(hg)_z = h_{gz} \circ g_z$  becomes the multiplier identity

$$J_{hg}(z) = J_g(z)J_h(gz). \quad (1.2)$$

For the induced automorphism of  $E^*$ , the place of  $J_g(z)$  is taken by  $J_g(z)^{*^{-1}}$ .

The sections of  $E$  (resp  $E^*$ ) in the trivialization become the holomorphic (resp anti-holomorphic) functions  $\hat{f}(z) = \varphi_z(f(z))$  (resp  $\psi_z(f(z))$ ). The action  $g^*f$  of an automorphism  $g$  on a section becomes  $(g^*\hat{f})(z) = J_g(z)\hat{f}(gz)$ . If  $G$  is a group of automorphisms of  $E$ , the representation  $U$  of  $G$  described in Section 1.1 becomes the “multiplier representation”

$$(\hat{U}\hat{f})(z) = J_{g^{-1}}(z)\hat{f}(g^{-1}z). \quad (1.3)$$

A Hermitian structure on  $E$  becomes a family of inner products on  $\mathbb{C}^n$ , parametrized by  $z \in M$ . One can always write

$$\|\xi\|_{E_z}^2 = \langle H(z)\xi, \xi \rangle_{\mathbb{C}^n}$$

with a positive definite matrix  $H(z)$ ,  $z \in M$ . The structure is invariant under a bundle automorphism  $\hat{g}$  if and only if  $\langle H(gz)J_g(z)^{-1}\xi, J_g(z)^{-1}\xi \rangle_{\mathbb{C}^n} = \langle H(z)\xi, \xi \rangle_{\mathbb{C}^n}$ , that is,

$$H(gz) = J_g(z)^*H(z)J_g(z).$$

The dual Hermitian structure of  $E^*$  is given by  $\|\xi\|_{E_z^*}^2 = \langle H(z)^{-1}\xi, \xi \rangle_{\mathbb{C}^n}$ .

A Hilbert space  $\mathcal{H}$  of sections of  $E$  becomes a space  $\hat{\mathcal{H}}$  of holomorphic functions from  $M$  to  $\mathbb{C}^n$ . The reproducing kernel becomes  $\hat{K}(z, w) = \varphi_z \circ K(z, w) \circ \psi_w^{-1}$ , a matrix valued function, holomorphic in  $z$  and anti-holomorphic in  $w$ . The reproducing property appears as

$$\langle \hat{f}(z), \xi \rangle_{\mathbb{C}^n} = \langle \hat{f}, \hat{K}_z \xi \rangle_{\hat{\mathcal{H}}},$$

the positivity as

$$\sum_{j,k} \langle \hat{K}(z_j, z_k) \xi_k, \xi_j \rangle_{\mathbb{C}^n} \geq 0,$$

and the isometry of the  $G$  - action as

$$J_g(z)\hat{K}(gz, gw)J_g(w)^* = \hat{K}(z, w).$$

The canonical Hermitian structure of  $E$  is then given by  $H(z) = K(z, z)^{-1}$ .

**1.5. Induced representations.** We briefly recall some known facts of representation theory. Let  $G, H$  be real (or, complex) Lie groups and  $H \subseteq G$  be closed. Given a representation  $\varrho$  of  $H$  on a complex finite dimensional vector space  $V$ , let  $\mathcal{F}(G, V)^H$  denote the linear space of  $C^\infty$  (or holomorphic) functions  $F : G \rightarrow V$  satisfying

$$F(gh) = \varrho(h)^{-1}F(g), \quad g \in G, h \in H.$$

The induced representation (cf. [5, p. 187])  $\mathbb{U} := \text{Ind}_H^G(\varrho)$  acts on the linear space  $\mathcal{F}(G, V)^H$  by left translation:  $(\mathbb{U}_{g_1}f)(g_2) = f(g_1^{-1}g_2)$ .

From the linear representations  $(\varrho, V)$  of  $H$ , one obtains all the  $G$  - homogeneous vector bundles over  $M = G/H$  as  $G \times_H V$ , which is  $(G \times V)/\sim$ , where

$$(gh, v) \sim (g, \varrho(h)v), \quad h, g \in G, v \in V.$$

The map  $(g, v) \mapsto gH$  is clearly constant on the equivalence class  $[(g, v)]$  and hence defines a map  $\pi : G \times_H V \rightarrow M$ . An action  $\hat{g}$ ,  $g \in G$ , of the group  $G$  is now defined on  $G \times_H V$  by setting  $\hat{g}'([(g, v)]) = [(g'g, v)]$ . This definition is independent of the choice of the representatives chosen. Thus  $G \times_H V$  is a homogeneous vector bundle on  $M$ . As in Section 1.1, there is a representation  $U$  of  $G$  on the sections of  $G \times_H V$ , where  $(U(g)s)(x) = \hat{g}(s(g^{-1} \cdot x))$ . The lift to  $G$  of the section  $s$  of the vector bundle  $G \times_H V$  is  $\tilde{s} : G \rightarrow V$  with  $\tilde{s}(g) := \hat{g}^{-1}s(gH)$ . These again form the space  $\mathcal{F}(G, V)^H$  which shows that  $U$  is just another realization of the representation  $\mathbb{U}$ .

When  $M$  is a manifold with a group  $G$  acting on it transitively, we use the usual identification  $M = G/H$ , where  $H$  is the stabilizer in  $G$  of a chosen fixed point  $0 \in M$ . The map  $q : g \mapsto g \cdot 0$  is the quotient map from  $G$  to  $M$ . Suppose that there exists a global cross-section  $p : M \rightarrow G$ , that is, a map with  $q \circ p = \text{id}_M$ . Then  $p$  gives a trivialization of the bundle  $E = G \times_H V$ . The trivializing map  $\varphi$  is given for  $v \in E_z$  by  $\varphi(v) = (z, p(z)^{-1}v)$ , that is,  $\varphi_z = p(z)^{-1}$ . (This  $\varphi$  actually maps to  $M \times E_0$ , but  $E_0$  with  $H$  acting on it by the bundle action can be identified with  $(\varrho, V)$ .) As in Section 1.4, the action of  $G$  on  $E_z$  becomes  $J_g(z)^{-1} = \varphi_{gz} \circ g_z \circ \varphi_z^{-1}$  which is now the group product  $p(gz)^{-1}gp(z)$  (preserving the fibre  $E_0$ ) followed by the identification of  $E_0$  with  $V = \mathbb{C}^n$ , that is,

$$J_g(z) = \varrho(p(z)^{-1}g^{-1}p(g(z))), \quad z \in M, g \in G. \quad (1.4)$$

The representation  $U$  appears now as the multiplier representation with multiplier (1.4).

Even though not needed in this paper, we point out that given any  $J : G \times M \rightarrow \text{GL}_n(\mathbb{C})$  satisfying the cocycle condition (1.2), the map  $(U_g f)(z) = J_g^{-1}(z)f(g^{-1} \cdot z)$  defines a multiplier representation of the group  $G$ . Also, it defines a representation  $\varrho : h \mapsto J_{h^{-1}}(0)$  of the group  $H$  on the vector space  $V$ . The representation induced by this  $\varrho$  is equivalent to  $U$ . In fact, the multiplier corresponding to the cross section  $p$  and the representation  $\varrho$  is

$$\begin{aligned} \varrho(p(z)^{-1}g^{-1}p(g \cdot z)) &= J_{p(g \cdot z)^{-1}gp(z)}(0) \\ &= J_{p(z)}(0)J_{p(g \cdot z)^{-1}g}(p(z) \cdot 0) \\ &= J_{p(z)}(0)J_g(z)J_{p(g \cdot z)^{-1}}(g \cdot z) \\ &= J_{p(z)}(0)J_g(z)J_{p(g \cdot 0)}(0)^{-1}. \end{aligned}$$

which is equivalent to the multiplier  $J$ .

We remark that the inducing construction always gives a multiplier such that  $J_g(z) \in \varrho(H)$  for all  $g, z$ . Not all multipliers possess this additional property. However, given any multiplier  $J$ , we can always find another multiplier  $J'$  equivalent to  $J$  such that  $J'_g(z) \in \varrho(H)$ , where  $\varrho(h) = J_{h^{-1}}(0)$ . This is achieved by taking any section  $p$  and setting

$$J'_g(z) = J_{p(z)}(0)J_g(z)J_{p(g \cdot z)}(0)^{-1}.$$

Holomorphic induced representation is a refinement of the induced representation in the case of real groups  $G, H$  such that  $G/H$  has a  $G$ -invariant complex structure. The complex structure determines a subalgebra  $\mathfrak{b}$  of  $\mathfrak{g}^{\mathbb{C}}$ , namely the isotropy algebra in the local action of  $\mathfrak{g}^{\mathbb{C}}$  on  $G/H$ . The holomorphic induced representation is the restriction of the induced representation to a subspace of  $\mathcal{F}(G, V)^H$ , defined by the differential equations  $XF = -\varrho(X)F$  for all  $X \in \mathfrak{b}$ , where  $\varrho$  now is a representation of the pair  $(H, \mathfrak{b})$ . It is an important fact that every  $G$ -homogeneous holomorphic vector bundle arises by holomorphic induction from a simultaneous finite dimensional representation  $\varrho$  of  $H$  and  $\mathfrak{b}$  (cf. [5, Ch. 13]). We will use this fact to determine all the holomorphic vector bundles which are homogeneous under the universal cover of the Möbius group.

## 2. HOMOGENEOUS HOLOMORPHIC VECTOR BUNDLES

In the following, we explicitly construct all the irreducible homogeneous holomorphic Hermitian vector bundles over the unit disc  $\mathbb{D}$ . Every homogeneous holomorphic Hermitian vector bundle on  $\mathbb{D}$  is then obtained as a direct sum of the irreducible ones (Corollary 2.1). In Section 4, we determine which ones of these irreducible homogeneous holomorphic Hermitian vector bundles over  $\mathbb{D}$  correspond to operators in the Cowen-Douglas class  $B_n(\mathbb{D})$ .

**2.1. The Möbius group.** Let  $G_0$  be the Möbius group – the group of bi-holomorphic automorphisms of the unit disc  $\mathbb{D}$ ,  $G = \text{SU}(1, 1)$  and  $\mathbb{K} \subseteq G$  be the rotation group. Let  $\tilde{G}$  be the universal covering group of  $G$  (and also that of the group  $G_0$ ). The group  $G$  acts on the unit disc  $\mathbb{D}$  according to the

rule

$$g : z \mapsto (az + b)(\bar{b}z + \bar{a})^{-1}, \quad g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in G, \quad z \in \mathbb{D}.$$

The group  $\tilde{G}$  also acts on  $\mathbb{D}$  (by  $g \cdot z = q(g) \cdot z$ , where  $q : \tilde{G} \rightarrow G$  is the covering map), we denote the stabilizer of 0 in it by  $\tilde{\mathbb{K}}$ . So  $\mathbb{D} \cong G/\mathbb{K} \cong \tilde{G}/\tilde{\mathbb{K}}$ . The complexification  $G^{\mathbb{C}}$  of the group  $G$  is the (simply connected) group  $\mathrm{SL}(2, \mathbb{C})$ .

In the following, we use the notation of [7, 9], which is the notation used in [13]. The Lie algebra  $\mathfrak{g}$  of the group  $G$  is spanned by  $X_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ,  $X_0 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$  and  $Y = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . The subalgebra  $\mathfrak{k}$  corresponding to  $\mathbb{K}$  is spanned by  $X_0$ . In the complexified Lie algebra  $\mathfrak{g}^{\mathbb{C}}$ , we mostly use the complex basis  $h, x, y$  given by

$$\begin{aligned} h &= -iX_0 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ x &= X_1 + iY = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ y &= X_1 - iY = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

The subgroup of  $G^{\mathbb{C}}$  corresponding to  $\mathfrak{g}$  is  $G$ . The group  $G^{\mathbb{C}}$  has the closed subgroups  $\mathbb{K}^{\mathbb{C}} = \left\{ \begin{pmatrix} z & 0 \\ 0 & \frac{1}{z} \end{pmatrix} : z \in \mathbb{C}, z \neq 0 \right\}$ ,  $P^+ = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{C} \right\}$ ,  $P^- = \left\{ \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} : z \in \mathbb{C} \right\}$ ; the corresponding Lie algebras  $\mathfrak{k}^{\mathbb{C}} = \left\{ \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} : c \in \mathbb{C} \right\}$ ,  $\mathfrak{p}^+ = \left\{ \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} : c \in \mathbb{C} \right\}$ ,  $\mathfrak{p}^- = \left\{ \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} : c \in \mathbb{C} \right\}$  are spanned by  $h, x$  and  $y$ , respectively. The product  $\mathbb{K}^{\mathbb{C}}P^- = \left\{ \begin{pmatrix} a & 0 \\ b & \frac{1}{a} \end{pmatrix} : 0 \neq a \in \mathbb{C}, b \in \mathbb{C} \right\}$  is a closed subgroup to be also denoted  $B$ ; its Lie algebra is  $\mathfrak{b} = \mathbb{C}h + \mathbb{C}y$ . The product set  $P^+\mathbb{K}^{\mathbb{C}}P^- = P^+B$  is dense open in  $G^{\mathbb{C}}$ , contains  $G$ , and the product decomposition of each of its elements is unique. ( $G^{\mathbb{C}}/B$  is the Riemann sphere,  $g\mathbb{K} \rightarrow gB$ , ( $g \in G$ ) is the natural embedding of  $\mathbb{D} \cong G/\mathbb{K}$  into it.) Linear representations  $(\varrho, V)$  of the algebra  $\mathfrak{b} \subseteq \mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C})$ , that is, pairs of linear transformations  $\varrho(h), \varrho(y)$  satisfying

$$[\varrho(h), \varrho(y)] = -\varrho(y) \tag{2.1}$$

are automatically representations of  $\mathbb{K}$  as well. Therefore they give, by holomorphic induction, all the homogeneous holomorphic vector bundles.

A homogeneous holomorphic vector bundle that admits a  $\tilde{G}$ -invariant Hermitian structure will be called *Hermitizable*. Since the vector bundles corresponding to operators in the Cowen - Douglas class are of this type, we only consider these bundles (except for some comments following Remark 3.1). The  $\tilde{G}$ -invariant Hermitian structures on the homogeneous holomorphic vector bundle (making it into a homogeneous holomorphic Hermitian vector bundle), if they exist, are given by  $\varrho(\tilde{\mathbb{K}})$ -invariant inner products on the representation space  $V$ . A  $\varrho(\tilde{\mathbb{K}})$ -invariant inner product exists if and only if  $\varrho(h)$  is diagonal with real diagonal elements in an appropriate basis. So, we will assume without restricting generality, that the representation space of  $\varrho$  is  $\mathbb{C}^d$  and that  $\varrho(h)$  is a real diagonal matrix.

Furthermore, we will be interested only in irreducible homogeneous holomorphic Hermitian vector bundles, this corresponds to  $\varrho$  not being the orthogonal direct sum of non-trivial representations.

Let  $V_\lambda$  be the eigenspace of  $\varrho(h)$  with eigenvalue  $\lambda$ . We say that a Hermitizable homogeneous holomorphic vector bundle is *elementary* if the eigenvalues of  $\varrho(h)$  form an uninterrupted string:  $-\eta, -(\eta+1), \dots, -(\eta+m)$ . Every irreducible homogeneous holomorphic Hermitian vector bundle is elementary. In fact, let  $-\eta$  be the largest eigenvalue of  $\varrho(h)$  and  $m$  be the largest integer such that  $-\eta, -(\eta+1), \dots, -(\eta+m)$  are all eigenvalues. From (2.1) we have  $\varrho(y)V_\lambda \subseteq V_{\lambda-1}$ ; this and orthogonality of the eigenspaces imply that  $V = \oplus_{j=0}^m V_{-(\eta+j)}$  and its orthocomplement are invariant

under  $\varrho$ . So,  $V$  is the whole space  $\mathbb{C}^d$ , and we have proved that the bundle is elementary. We can write  $V_{(\eta+j)} = \mathbb{C}^{d_j}$  and hence  $(\varrho, \mathbb{C}^d)$  is described by the two matrices:

$$\varrho(h) = \begin{pmatrix} -\eta I_0 & & \\ & \ddots & \\ & & -(\eta+m)I_m \end{pmatrix},$$

where  $I_j$  is the identity matrix on  $\mathbb{C}^{d_j}$  and

$$Y := \varrho(y) = \begin{pmatrix} 0 & & & & \\ Y_1 & 0 & & & \\ & Y_2 & 0 & & \\ & & \ddots & \ddots & \\ & & & Y_m & 0 \end{pmatrix}$$

for some choice of matrices  $Y_1, \dots, Y_m$  that represent the linear transformations  $Y_j : \mathbb{C}^{d_{j-1}} \rightarrow \mathbb{C}^{d_j}$ . Let  $E^{(\eta, Y)}$  denote the holomorphic bundle induced by this representation.

It is clear that  $\varrho$  can be written as the tensor product of the one dimensional representation  $\sigma$  given by  $\sigma(h) = -\eta$ ,  $\sigma(y) = 0$ , and the representation  $\varrho^0$  given by  $\varrho^0(h) = \varrho(h) + \eta I$ ,  $\varrho^0(y) = \varrho(y)$ . Correspondingly, the bundle  $E^{(\eta, Y)}$  for  $\varrho$  is the tensor product of a line bundle  $L_\eta$  and the bundle corresponding to  $\varrho^0$ , that is,  $E^{(\eta, Y)} = L_\eta \otimes E^{(0, Y)}$ .

For  $g \in \tilde{G}$ ,  $g'(z)$  (we write  $g'(z) = \frac{\partial g}{\partial z}(z)$ ) is a real analytic function on the simply connected set  $\tilde{G} \times \mathbb{D}$ , holomorphic in  $z$ . Also  $g'(z) \neq 0$  since  $g$  is one-one and holomorphic. Given any  $\lambda \in \mathbb{R}$ , taking the principal branch of the power function when  $g$  is near the identity, we can uniquely define  $g'(z)^\lambda$  as a real analytic function on  $\tilde{G} \times \mathbb{D}$  which is holomorphic on  $\mathbb{D}$  for all fixed  $g \in \tilde{G}$ .

For the line bundle  $L_\eta$ , the multiplier is  $g'(z)^\eta$ . Consequently, the multiplier corresponding to the original  $\varrho$  is

$$J_g(z) = (g'(z))^\eta J_g^0(z), \quad (2.2)$$

where  $J^0$  is the multiplier obtained from  $\varrho^0$ .

The advantage of  $\varrho^0$  is that it is also a representation of  $G$  (not only of  $\tilde{G}$ ) and extends to a representation of  $G^\mathbb{C}$ . The (ordinary) induced representation (in the holomorphic category)  $\text{Ind}_T^G(\varrho)$  operates on functions  $F : G^\mathbb{C} \rightarrow V$  such that  $F(gt) = \varrho^0(t)^{-1}F(g)$  ( $g \in G^\mathbb{C}$ ,  $t \in T$ ). The restrictions of these functions  $F$  to  $G$  then give exactly the functions  $\Phi : G \rightarrow V$  which satisfy  $\Phi(gk) = \varrho^0(k)^{-1}\Phi(g)$  ( $g \in G^\mathbb{C}$ ,  $t \in T$ ) and  $(X\Phi)(g) = -\varrho^0(X)\Phi(g)$  ( $g \in G$ ,  $X \in \mathfrak{b}$ ), that is, the space of the representation holomorphically induced by  $\varrho^0$ . Taking a holomorphic local cross section  $p$  of  $G^\mathbb{C}$  defined on  $\mathbb{D}$ , the functions  $f(z) = F(p(z))$  give a trivialization of  $E^{(0, Y)}$ .

We use the local cross section  $p : \mathbb{D} \rightarrow G^\mathbb{C}$ ,  $z \mapsto p(z) := \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$ . Apply (1.4) to compute the corresponding multiplier  $J_g^0(z)$ . For  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ , we have

$$\begin{aligned} J_g^0(z) &= \varrho^0 \left( \begin{pmatrix} 1 & -g \cdot z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \right)^{-1} \\ &= \varrho^0 \begin{pmatrix} cz + d & 0 \\ -c & (cz + d)^{-1} \end{pmatrix} \\ &= \varrho^0 \left( \begin{pmatrix} (cz + d)^{\frac{1}{2}} & 0 \\ 0 & (cz + d)^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix} \begin{pmatrix} (cz + d)^{\frac{1}{2}} & 0 \\ 0 & (cz + d)^{-\frac{1}{2}} \end{pmatrix} \right) \\ &= \varrho^0(\exp(2 \log(cz + d)^{\frac{1}{2}} h)) \varrho^0(\exp(-cy)) \varrho^0(\exp(2 \log(cz + d)^{\frac{1}{2}} h)) \\ &= D_g(z) \exp(-cY) D_g(z), \end{aligned} \quad (2.3)$$

where  $D_g(z)$  is the diagonal matrix with

$$D_g(z)_{\ell\ell} = (cz + d)^{-\frac{i}{2}} I_{d_j}.$$

Computing the matrix entries of the exponential using (2.2), we obtain for  $g \in \tilde{G}$ ,  $z \in \mathbb{D}$ ,

$$\begin{aligned} (J_g^{(\eta, Y)}(z))_{p, \ell} &:= (g'(z))^\eta J_g^0(z) \\ &= \begin{cases} \frac{1}{(p-\ell)!} (-c_g)^{p-\ell} g'(z)^{\eta + \frac{p+\ell}{2}} Y_p \cdots Y_{\ell+1} & \text{if } p \geq \ell \\ 0 & \text{if } p < \ell. \end{cases} \end{aligned} \quad (2.4)$$

In this formula  $c_g$  for  $g \in \tilde{G}$  is to be understood as  $c_{g^\#}$ , where  $g^\#$  is the projection to  $G$  of  $g$ . We note here, for later use, that there is also another way to interpret  $c_g$  for  $g \in \tilde{G}$ . Taking a small neighborhood  $\tilde{U}$  of the identity in  $\tilde{G}$  such that the projection is a diffeomorphism onto a neighborhood  $U$  of the identity in  $G$ , by computing in  $U$ , we find that

$$g''(z) = -2c_g g'(z)^{3/2} \quad (2.5)$$

holds with  $c_g$  an analytic function of  $g$  on  $U$ , independent of  $z$ . This is then true for  $g \in \tilde{U}$  and by analytic continuation for all  $g \in \tilde{G}$ . So the equation (2.5) can serve as a definition for  $c_g$ .

**PROPOSITION 2.1.** *All elementary Hermitizable homogeneous holomorphic vector bundles are of the form  $E^{(\eta, Y)}$  with  $\eta \in \mathbb{R}$  and  $Y$  as before. The bundles  $E^{(\eta, Y)}$  and  $E^{(\eta', Y')}$  are isomorphic if and only if  $\eta = \eta'$  and  $Y' = AY A^{-1}$  with a block diagonal matrix  $A$ .*

*Proof.* The induced bundles are isomorphic if and only if the inducing representations  $\varrho, \varrho'$  are linearly equivalent, that is,  $\varrho' = A\varrho A^{-1}$  for some  $A$ . Since we have normalized the representations by fixing the matrix  $\varrho(h)$ , the equivalence must be given by an  $A$  which commutes with  $\varrho(h)$ , that is, by a block diagonal  $A$ .  $\square$

Thus  $E^{(\eta, \{Y\})} = L_\eta \otimes E^{(\{Y\})}$  parametrizes the equivalence classes of elementary Hermitizable homogeneous holomorphic vector bundles. Here, we have let  $\{Y\}$  denote the conjugacy class of  $Y$  under conjugation by a block diagonal matrix  $A$ .

**2.2. Homogeneous holomorphic Hermitian vector bundles.** We proceed to discuss homogeneous holomorphic Hermitian vector bundles. From here on we will always use the trivialization we just described. We will not always make a careful distinction between a section of  $E^{(\eta, Y)}$  and the functions from  $\mathbb{D}$  to  $\mathbb{C}^d$  on which  $G$  acts by the multiplier  $J_g^{(\eta, Y)}(z)$ . As in Section 1, a Hermitian structure appears in the trivialization as a family of quadratic forms  $\langle H(z)\xi, \xi \rangle$ , which because of the homogeneity is determined by a single positive definite block-diagonal matrix  $H = H(0)$ . We denote by  $(E^{(\eta, Y)}, H)$  the bundle  $E^{(\eta, Y)}$  equipped with the Hermitian structure determined by  $H$ .

**PROPOSITION 2.2.** *The Hermitian vector bundles  $(E^{(\eta, Y)}, H)$  and  $(E^{(\eta', Y')}, H')$  are isomorphic if and only if  $\eta = \eta'$ ,  $Y' = AY A^{-1}$  and  $H' = A^{*-1} H A$  with a block diagonal matrix  $A$ .*

*Proof.* The trivialization of the sections obtained by starting with  $\varrho$  (resp.  $\varrho' = A\varrho A^{-1}$ ) are related as  $f'(z) = Af(z)$ . Now,  $H'(z)$  gives the same metric as  $H(z)$  if and only if  $\langle H'(z)f'(z), f'(z) \rangle = \langle H(z)f(z), f(z) \rangle$ . From this, the statement follows.  $\square$

For any  $H$ , clearly there is an  $A$  such that  $A^{*-1} H A = I$ . This means that every elementary homogeneous holomorphic Hermitian vector bundle is isomorphic to one of the form  $(E^{(\eta, Y)}, I)$ . Two vector bundles of this form are equivalent if and only if  $Y' = AY A^{-1}$  with  $A$  such that  $A^{*-1} I A^{-1} = I$ , that is, with a block-diagonal unitary  $A$ . We denote by  $[Y]$  the equivalence class of  $Y$  under conjugation by block-diagonal unitaries and write  $E^{(\eta, [Y])}$  for the equivalence class of  $(E^{(\eta, Y)}, I)$ , omitting the  $I$ . We now have the first half of the following Proposition.

**PROPOSITION 2.3.** *For  $\eta \in \mathbb{R}$ ,  $[Y]$  a block unitary conjugacy class of matrices  $Y$ , the vector bundles  $E^{(\eta, [Y])}$  form a parametrization of the elementary homogeneous holomorphic Hermitian vector bundles. The Hermitian vector bundle  $E^{(\eta, [Y])}$  is irreducible if and only if  $Y$  cannot be split into orthogonal direct sum  $Y' \oplus Y''$  with  $Y', Y''$  having the same block diagonal form as  $Y$ .*

*Proof.* The last statement follows since the irreducibility of  $E^{(\eta, [Y])}$  is the same as the possibility of splitting  $\varrho$  into an orthogonal direct sum of two sub-representations.  $\square$

Proposition 2.3, with a different proof, also appears in [3].

The following Theorem is important because its hypothesis is exactly what we know about the vector bundle corresponding to a homogeneous operator in the Cowen-Douglas class  $B_n(\mathbb{D})$ . It was stated in [7] but proved without the uniqueness statement. Here we give a complete proof.

**THEOREM 2.1.** *Let  $E$  be a Hermitian holomorphic vector bundle over  $\mathbb{D}$  and suppose that for all  $g \in G$ , there exists an automorphism of  $E$  whose action on  $\mathbb{D}$  coincides with  $g$ . Then the full automorphism group of  $E$  is reductive and  $\tilde{G}$  acts on  $E$  by automorphisms in a unique way.*

*Proof.* Let  $\hat{G}$  denote the connected component of the automorphism group of  $E$ . It is a Lie group because it is the connected component of the isometry group of  $E$  under the Riemannian metric defined for vectors tangent to the fibres by the Hermitian structure and for vectors horizontal with respect to the Hermitian connection by the  $G$ -invariant metric of  $\mathbb{D}$ .

Let  $N \subseteq \hat{G}$  be the subgroup of elements acting on  $\mathbb{D}$  as the identity map. The subgroup  $N$  is normal, and the projection  $\pi : \hat{G} \rightarrow G$  is a homomorphism with kernel  $N$ . Let  $\mathbb{K}$  be the stabilizer of 0 in  $G$  and let  $\hat{\mathbb{K}} = \pi^{-1}(\mathbb{K})$ . The group  $\hat{\mathbb{K}}$  contains  $N$  and is compact because it is the stabilizer of the origin in the fiber over 0.

Let  $\hat{\mathfrak{g}}, \mathfrak{g}, \hat{\mathfrak{k}}, \mathfrak{n}, \hat{\mathfrak{k}}$  be the Lie algebras of the groups defined above, and let  $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$  be the Cartan decomposition. Since  $\hat{\mathbb{K}}$  is compact, we can choose an  $\text{Ad}(\hat{\mathbb{K}})$ -invariant complement  $\hat{\mathfrak{p}}$  to  $\hat{\mathfrak{k}}$  in  $\hat{\mathfrak{g}}$ . Now,  $\pi_*$  maps  $\hat{\mathfrak{k}}$  onto  $\mathfrak{k}$  with kernel  $\mathfrak{n}$ . By counting dimension, it follows that  $\pi_*$  maps  $\hat{\mathfrak{p}}$  to  $\mathfrak{p}$  bijectively.

We set  $\hat{\mathfrak{k}}_0 = [\hat{\mathfrak{p}}, \hat{\mathfrak{p}}]$ . Then  $\pi_*(\hat{\mathfrak{k}}_0) = [\pi_*\hat{\mathfrak{p}}, \pi_*\hat{\mathfrak{p}}] = \mathfrak{k}$ , therefore  $\hat{\mathfrak{k}}_0 \subseteq \pi_*^{-1}(\mathfrak{k}) = \hat{\mathfrak{k}}$ . It follows that  $[\hat{\mathfrak{k}}_0, \hat{\mathfrak{p}}] \subseteq \hat{\mathfrak{p}}$  and by the Jacobi identity,  $\hat{\mathfrak{g}}_0 = \hat{\mathfrak{k}}_0 + \hat{\mathfrak{p}}$  is a subalgebra. Similarly,  $[\mathfrak{n}, \hat{\mathfrak{p}}] \subseteq \hat{\mathfrak{p}}$  since  $\mathfrak{n} \subseteq \hat{\mathfrak{k}}$ . But  $\mathfrak{n}$  is an ideal, so  $[\mathfrak{n}, \hat{\mathfrak{p}}] = 0$ , and by the Jacobi identity  $[\mathfrak{n}, \hat{\mathfrak{g}}_0] = 0$ . Finally,  $\hat{\mathfrak{g}} = \mathfrak{n} \oplus \hat{\mathfrak{g}}_0$  and  $\mathfrak{g}$  is reductive. The analytic subgroup  $\hat{G}_0 \subseteq \hat{G}$  corresponding to  $\hat{\mathfrak{g}}_0$  is a covering group of  $G$  and therefore it acts on  $E$  by automorphisms. It is the unique subgroup of  $\hat{G}$  with this property because  $\hat{\mathfrak{g}}_0$ , being the maximal semisimple ideal in the reductive algebra  $\hat{\mathfrak{g}}$ , is uniquely determined. The action of  $\hat{G}_0$  now lifts to a unique action of  $\tilde{G}$ .  $\square$

Theorem 2.1 implies that every homogeneous operator in the Cowen-Douglas class  $B_n(\mathbb{D})$  has an associated representation irrespective of whether it is irreducible or not. The following Corollary has also appeared in [3].

**COROLLARY 2.1.** *If a Hermitian holomorphic vector bundle  $E$  is homogeneous and is reducible ( $E = E_1 \oplus E_2$ ) as a Hermitian holomorphic vector bundle then it is reducible as a homogeneous Hermitian holomorphic vector bundle, that is,  $E_1$  and  $E_2$  are also homogeneous.*

*Proof.* We consider the automorphisms  $\exp th$  of  $E$ , where

$$h = \begin{cases} iI & \text{on } E_1 \\ -iI & \text{on } E_2. \end{cases}$$

Then  $h$  is in  $\mathfrak{n}$  since  $\exp th$  ( $t \in \mathbb{R}$ ) preserves fibres. So,  $h$  commutes with  $\hat{\mathfrak{g}}_0$ . The sections of  $E_1, E_2$  are characterized as eigensections of  $h$  corresponding to different eigenvalues. Thus  $\hat{G}_0$ , and its universal covering  $\tilde{G}$  preserve the eigensections of  $h$ .  $\square$

### 3. HOMOGENEOUS HOLOMORPHIC HERMITIAN VECTOR BUNDLES WITH REPRODUCING KERNEL

In this Section, we determine which ones of the elementary homogeneous holomorphic Hermitian vector bundles have their Hermitian structure coming from a reproducing kernel. In other words, which are the homogeneous holomorphic vector bundles that have a  $\tilde{G}$ -invariant reproducing kernel  $K(z, w)$ . When there is a reproducing kernel  $K$ , it gives a canonical Hermitian structure by setting  $H = K(0, 0)^{-1}$ . Let  $p_z = \frac{1}{\sqrt{1-|z|^2}} \begin{pmatrix} 1 & z \\ \bar{z} & 1 \end{pmatrix} \in G$ , so  $p_z \cdot 0 = z$ . Writing  $J_z^{(\eta, Y)}$  for  $J_{p_z}^{(\eta, Y)}(z)$ , we have

$$K(z, z) = J_z^{(\eta, Y)} K(0, 0) J_z^{(\eta, Y)*}. \quad (3.1)$$

So, the question amounts to enumeration of all the possibilities for  $K(0, 0)$ .

**3.1. An intertwining map.** For  $\lambda > 0$ , let  $\mathbb{A}^{(\lambda)}$  be the Hilbert space of holomorphic functions on the unit disc with reproducing kernel  $(1 - z\bar{w})^{-2\lambda}$ . It corresponds to the homogeneous line bundle  $L_\lambda$ . The group  $\tilde{G}$  acts on it unitarily with the multiplier  $g'(z)$ . This action is the Discrete series representation  $D_g^{(\lambda)}$ . Let  $\mathbb{C}^d = \bigoplus_{j=0}^d \mathbb{C}^{d_j}$ . We think of functions  $f : \mathbb{D} \rightarrow \mathbb{C}^d$  as having components  $f_j : \mathbb{D} \rightarrow \mathbb{C}^{d_j}$ . Let  $\mathbf{A}^{(\eta)} = \bigoplus_{j=0}^m \mathbb{A}^{(\eta+j)} \otimes \mathbb{C}^{d_j}$ . For  $\eta > 0$ ,  $Y$  as before and  $f_j \in \mathbb{A}^{(\eta+j)} \otimes \mathbb{C}^{d_j}$ , define

$$(\Gamma^{(\eta, Y)} f_j)_\ell = \begin{cases} \frac{1}{(\ell-j)!} \frac{1}{(2\eta+2j)_{\ell-j}} Y_\ell \cdots Y_{j+1} f_j^{(\ell-j)} & \text{if } \ell \geq j \\ 0 & \text{if } \ell < j. \end{cases} \quad (3.2)$$

So,  $\Gamma^{(\eta, Y)}$  maps  $\text{Hol}(\mathbb{D}, \mathbb{C}^d)$  into itself. Let  $N$  be an invertible  $d \times d$  block diagonal matrix with blocks  $N_j$ ,  $0 \leq j \leq m$ ,  $d = d_0 + \cdots + d_m$ . We will assume throughout that  $N_0 = I_{d_0}$ . This is only a normalizing condition. We can normalize further by assuming that each block diagonal matrix with  $d_j \times d_j$  blocks  $N_j$  is positive definite but this is not important. We can think of  $N$  as changing the natural inner product of each  $\mathbb{C}^{d_j}$  to  $\langle N_j u, N_j v \rangle_{\mathbb{C}^{d_j}}$ . We let  $\Gamma_N^{(\eta, Y)} = \Gamma^{(\eta, Y)} \circ N$  and  $\mathcal{H}_N^{(\eta, Y)}$  denote the image of  $\Gamma_N^{(\eta, Y)}$  in the space of holomorphic functions  $\text{Hol}(\mathbb{D}, \mathbb{C}^d)$ .

**THEOREM 3.1.** *The map  $\Gamma_N^{(\eta, Y)}$  is a  $\tilde{G}$ -equivariant isomorphism of  $\mathbf{A}^{(\eta)}$  onto the Hilbert space  $\mathcal{H}_N^{(\eta, Y)}$  on which the  $\tilde{G}$  action is unitary via the multiplier  $J_g^{(\eta, Y)}(z)$ . It has a reproducing kernel  $K_N^{(\eta, Y)}(z, w)$  such that*

$$(K_N^{(\eta, Y)}(0, 0))_{\ell\ell} = \sum_{j=0}^{\ell} \frac{1}{(\ell-j)!} \frac{1}{(2\eta+2j)_{\ell-j}} Y_\ell \cdots Y_{j+1} N_j N_j^* Y_{j+1}^* \cdots Y_\ell^*.$$

*Proof.* The injectivity of the map  $\Gamma_N^{(\eta, Y)}$  is clear from its definition. It is also apparent that the image  $\mathcal{H}_N^{(\eta, Y)}$  is the algebraic direct sum of the summands  $\mathbb{A}^{(\eta+j)} \otimes \mathbb{C}^{d_j}$  of  $\mathbf{A}^{(\eta)}$ . We define a norm on  $\mathcal{H}_N^{(\eta, Y)}$  by stipulating that  $\Gamma_N^{(\eta, Y)}$  is a Hilbert space isometry. This gives us the Hilbert space  $\mathcal{H}_N^{(\eta, Y)}$  and the unitary action  $U_g$  of  $\tilde{G}$  on it. We have to show that it is the multiplier action given by  $J_g^{(\eta, Y)}(z)$ . For this, we must verify that

$$\Gamma_N^{(\eta, Y)} \circ \left( \bigoplus d_j D_{g^{-1}}^{(\eta+j)} \right) = U_{g^{-1}} \circ \Gamma_N^{(\eta, Y)}. \quad (3.3)$$

Since  $N$  obviously intertwines  $\bigoplus d_j D_{g^{-1}}^{(\eta+j)}$  with itself, it suffices to prove (3.3) for  $\Gamma^{(\eta, Y)}$  in place of  $\Gamma_N^{(\eta, Y)} = \Gamma^{(\eta, Y)} \circ N$ . Furthermore, it is enough to verify this relation for each  $f \in \mathbb{A}^{(\eta+j)} \otimes \mathbb{C}^{d_j}$ , that is, to show

$$\Gamma^{(\eta, Y)}((g')^{\eta+j}(f \circ g)) = J_g((\Gamma^{(\eta, Y)} f) \circ g), \quad f \in \mathbb{A}^{(\eta+j)} \otimes \mathbb{C}^{d_j}, \quad 0 \leq j \leq m.$$

We will show that the  $\ell$ th components on both sides are equal. First, if  $\ell < j$  then both sides are 0. Second if  $\ell \geq j$ , on the one hand, using Lemma 3.1 of [7] which is easily proved by induction starting

from the equation (2.5) and says that

$$((g')^\ell(f \circ g))^{(k)} = \sum_{i=0}^k \binom{k}{i} (2\ell + i)_{k-i} (-c)^{k-i} (g')^{\ell + \frac{k+i}{2}} (f^{(i)} \circ g) \quad (3.4)$$

for any  $g \in \tilde{G}$ , we have

$$\begin{aligned} & \Gamma^{(\eta, Y)}((g')^{\eta+j}(f \circ g)) \\ &= \frac{1}{(\ell-j)!} \frac{1}{(2\eta+2j)_{\ell-j}} Y_\ell \cdots Y_{j+1} ((g')^{\eta+j}(f \circ g))^{(\ell-j)} \\ &= \frac{1}{(\ell-j)!} \frac{1}{(2\eta+2j)_{\ell-j}} Y_\ell \cdots Y_{j+1} \sum_{i=0}^{\ell-j} \binom{\ell-j}{i} (2\eta+2j+i)_{\ell-j-i} (-c)^{\ell-j-i} (g')^{\eta+j+\frac{\ell-j+i}{2}} (f^{(i)} \circ g) \\ &= Y_\ell \cdots Y_{j+1} \sum_{i=0}^{\ell-j} \frac{1}{(\ell-j-i)! i!} \frac{1}{(2\eta+2j)_i} (-c)^{\ell-j-i} (g')^{\eta+j+\frac{\ell-j+i}{2}} (f^{(i)} \circ g), \end{aligned}$$

On the other hand,

$$\begin{aligned} & \sum_{p=j}^m (J_g)_{\ell p} ((\Gamma^{(\eta, Y)} f)_p \circ g) \\ &= \sum_{p=j}^{\ell} (-c)^{\ell-p} \frac{1}{(\ell-p)!} (g')^{\eta+\frac{p+\ell}{2}} Y_\ell \cdots Y_{p+1} \frac{1}{(p-j)!} \frac{1}{(2\eta+2j)_{p-j}} Y_p \cdots Y_{j+1} f^{(p-j)} \circ g \\ &= \sum_{p=j}^{\ell} \frac{1}{(\ell-p)!} \frac{1}{(p-j)!} \frac{1}{(2\eta+2j)_{p-j}} (-c)^{\ell-p} (g')^{\eta+\frac{p+\ell}{2}} Y_\ell \cdots Y_{j+1} f^{(p-j)} \circ g \\ &= \sum_{i=0}^{\ell-j} \frac{1}{(\ell-i-j)!} \frac{1}{i!} \frac{1}{(2\eta+2j)_i} (-c)^{\ell-j-i} (g')^{\eta+\frac{i+j+\ell}{2}} Y_\ell \cdots Y_{j+1} f^{(p-j)} \circ g. \end{aligned}$$

This completes the verification of (3.3). Finally, we observe that  $\mathcal{H}_N^{(\eta, Y)}$  has a reproducing kernel  $K_N^{(\eta, Y)}(z, w)$  because it is the image of  $\mathbf{A}^{(\eta)}$  under an isomorphism given by a holomorphic differential operator, so point evaluations remain continuous. Then  $K_N^{(\eta, Y)}(z, w)$  is obtained by applying  $\Gamma_N^{(\eta, Y)}$  to the reproducing kernel of  $\mathbf{A}^{(\eta)}$  once as a function of  $z$  and once as a function of  $w$ . This computation is easily carried out and gives the formula for  $K_N^{(\eta, Y)}(0, 0)$ .  $\square$

Writing  $H := H_N^{(\eta, Y)} = (K_N^{(\eta, Y)}(0, 0))^{-1}$ , the Hilbert space  $\mathcal{H}_N^{(\eta, Y)}$  is a space of sections of the homogeneous holomorphic Hermitian vector bundle  $(E^{(\eta, Y)}, H)$  in our (canonical) trivialization.

**THEOREM 3.2.** *The construction with  $\Gamma_N^{(\eta, Y)}$  gives all elementary homogeneous holomorphic Hermitian vector bundles which possess a reproducing kernel, namely, those of the form*

$$(E^{(\eta, Y)}, (K_N^{(\eta, Y)}(0, 0))^{-1}),$$

where  $\eta > 0$ ,  $Y$  are arbitrary and  $K_N^{(\eta, Y)}(0, 0)$  is the form given in Theorem 3.1. The vector bundles  $(E^{(\eta, Y)}, (K_N^{(\eta, Y)}(0, 0))^{-1})$  and  $(E^{(\eta', Y')}, (K_{N'}^{(\eta', Y')}(0, 0))^{-1})$  are equivalent if and only if  $\eta = \eta'$ ,  $Y' = AY A^{-1}$  and  $N' N'^* = A N N^* A^*$  for some invertible block diagonal matrix  $A$  of size  $d \times d$ .

*Proof.* The existence of a reproducing kernel implies that the vector bundle is Hermitizable. Such a bundle is of the form  $(E^{(\eta, Y)}, H)$  by Propositions 2.1 and 2.2. When it has a reproducing kernel, then in our canonical trivialization this is a matrix valued function  $K(z, w)$ , and we have  $H = K(0, 0)^{-1}$ . The  $\tilde{G}$  action  $U$  which is now unitary, is given by the multiplier  $J_g^{(\eta, Y)}(z)$ . The equation (2.4) shows

that the action of  $\tilde{\mathbb{K}}$  is diagonalized by the polynomial vectors: If  $v_j \in \mathbb{C}^{d_j}$  and  $f(z) = z^n v_j$ , then for  $k_\theta$  such that  $k_\theta(z) = e^{i\theta} z$ , we have  $U_{k_\theta} f = e^{i\theta(\eta+j+k)} f$ . It follows that  $U$  is a direct sum of the Discrete series representations  $D^{(\eta+j)}$ ,  $0 \leq j \leq m$ . In particular, it follows that  $\eta > 0$ .

The map  $\Gamma^{(\eta,Y)}$  (and  $\Gamma_N^{(\eta,Y)}$  for any block diagonal  $N$ ) intertwines the representations  $U$  and  $\oplus_{j=0}^m d_j D^{(\eta+j)}$ , both of which are unitary. By Schur's Lemma it follows that  $N$  can be chosen such that  $\Gamma^{(\eta,Y)} \circ N$  is unitary. This proves that the bundle  $E^{(\eta,Y)}$  corresponds to the Hilbert space  $\mathcal{H}_N^{(\eta,Y)}$ .

The statement about equivalence follows from the analogous statement in Proposition 2.2.  $\square$

**REMARK 3.1.** *In the proof we only used the unitarizability of the  $\tilde{G}$  action on the sections of the Hermitizable bundle  $E$ . In this vein, an even more general result holds:*

*For any  $\tilde{G}$  - homogeneous holomorphic vector bundle  $E$ , if the  $\tilde{G}$  action on the sections is unitarizable then it is a direct sum of bundles corresponding to some  $\mathcal{H}_N^{(\eta,Y)}$ . (The possible unitary structures correspond to different choices of  $N$ .)*

One way to prove this is to use the ‘‘Inverse propagation theorem’’ of T. Kobayashi [6]. If the action of  $\tilde{G}$  is unitary, then so is the  $\tilde{\mathbb{K}}$  action on the fibres, and we are back in the situation of Theorem 3.2.

Here we sketch a more direct proof which also shows what the non-Hermitizable homogeneous holomorphic vector bundles are like.

A general  $E$  is still gotten from two matrices  $Z = \varrho(h)$ ,  $Y = \varrho(y)$  such that  $[Z, Y] = -Y$  by holomorphic induction. The inclusion  $YV_\lambda \subseteq V_{\lambda-1}$  still holds for the generalized eigenspaces of  $Z$ . Using some easy identities for  $g'(z)$ , we can then verify that

$$J_g(z) = \exp\left(\frac{1}{2}(\log(g'(z)))'Y\right) \exp\left(-\log g'(z)Z\right),$$

which, in the case where  $Z$  is diagonal, is just another way to write (3.2), is a multiplier.

Writing  $U_g$  for the action of  $\tilde{G}$  on  $\text{Hol}(D, V)$  given by  $J_g(z)$ , we compute

$$(U_{\exp tih} f)(z) = \exp(itZ) f(e^{-it}z). \quad (3.5)$$

Hence  $(U_h f)(0) = Zf(0)$  and by a similar computation  $(U_y f)(0) = Yf(0)$ . This shows that  $J_g(z)$  gives a trivialization of our  $E$ . It also shows that  $U_k$ ,  $k \in \tilde{\mathbb{K}}$  maps the spaces  $\mathcal{M}_p$  of monomials of degree  $p$  to  $\mathcal{M}_p$  for all  $p \geq 0$ . Hence  $\tilde{\mathbb{K}}$  - finite vectors are exactly the  $(V$  - valued) polynomials.

Now if  $U$  is unitary with respect to some inner product, then it is a sum of irreducible representations. The  $\tilde{\mathbb{K}}$ -types of these (i.e. the eigenfunctions of  $U_h$ ) are known to be one dimensional and together they span the space of  $\tilde{\mathbb{K}}$  - finite vectors. By (3.5),  $U_h$  maps any  $z^p v \in \mathcal{M}_p$  to  $z^p(Zv - pv)$ . It follows that  $Z$  must be diagonalizable, otherwise the eigenfunctions of  $U_h$  could not span  $\mathcal{M}_p$ .

**3.2. Parametrization.** The description of the homogeneous holomorphic Hermitian vector bundles given in Theorem 3.2 can be made more explicit. We now proceed to determine, in terms of the parametrization  $E^{(\eta,[Y])}$  of elementary homogeneous holomorphic Hermitian vector bundles as in Proposition 2.3, exactly which ones of these have their Hermitian structure come from a reproducing kernel.

**THEOREM 3.3.** *The Hermitian structure of  $E^{(\eta,[Y])}$  comes from a  $(\tilde{G}$  - invariant) reproducing kernel if and only if  $\eta > 0$  and*

$$I - Y_j \left( \sum_{k=0}^{j-1} \frac{(-1)^{j+k}}{(j-k)!(2\eta+j+k-1)_{j-k}} Y_{j-1} \cdots Y_{k+1} Y_{k+1}^* \cdots Y_{j-1}^* \right) Y_j^* > 0$$

for  $j = 1, 2, \dots, m$ .

*Proof.* We have a description of all the vector bundles with reproducing kernel in Theorem 3.2. To see how this description appears in the parametrization  $E^{(\eta,Y)}$ , we have to answer the question: For

what  $\eta, [Y]$ , is it possible to find a block-diagonal  $N$  such that  $K_N^{(\eta, Y)}(0, 0) = I$ . Writing this more explicitly, we have the system of equations

$$I_\ell - \sum_{j=0}^{\ell} \frac{1}{(\ell-j)!} \frac{1}{(2\eta+2j)_{\ell-j}} Y_\ell \cdots Y_{j+1} N_j N_j^* Y_{j+1}^* \cdots Y_\ell^* = 0, \quad (3.6)$$

$\ell = 1, \dots, m$  and the question is if the solution  $N_j N_j^*$ ,  $j = 1, \dots, m$  consists of positive definite matrices.

We claim that the solution of (3.6) is given by

$$N_j N_j^* = \sum_{k=0}^j \frac{(-1)^{j+k}}{(j-k)!(2\eta+j+k-1)_{j-k}} Y_j \cdots Y_{k+1} Y_{k+1}^* \cdots Y_j^*, \quad (3.7)$$

for  $j = 1, \dots, m$ .

In fact, substituting the expression for  $N_j N_j^*$  from (3.7) into (3.6), we have

$$I_\ell - \sum_{j=0}^{\ell} \sum_{k=0}^j \frac{1}{(\ell-j)!(2\eta+2j)_{\ell-j}} \frac{(-1)^{j+k}}{(j-k)!(2\eta+j+k-1)_{j-k}} Y(k) = 0,$$

where  $Y(k) = Y_\ell \cdots Y_{k+1} Y_{k+1}^* \cdots Y_\ell^*$ . The coefficient of  $Y(k)$ , from the above, is seen to be

$$\frac{1}{(\ell-k)!^2} \sum_{j=k}^{\ell} (-1)^{j+k} \binom{\ell-k}{j-k} (2\eta+2j-1) B(2\eta+k+j-1, \ell-k+1),$$

where  $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$  is the usual Beta function. Using the binomial formula and the integral representation:  $B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt$ , it simplifies to

$$\begin{aligned} & \frac{1}{(\ell-k)!^2} \int_0^1 \left\{ (2\eta+2k-1) t^{2\eta+2k-2} (1-t)^{2(\ell-k)} - 2(\ell-k) t^{2\eta+2k-1} (1-t)^{2(\ell-k)-1} \right\} dt \\ &= \frac{1}{(\ell-k)!^2} \int_0^1 \left\{ t^{2\eta+2k-2} (1-t)^{2(\ell-k)-1} ((2\eta+2k-1) - (2\eta+2\ell-1)t) \right\} dt \\ &= \frac{1}{(\ell-k)!^2} (xB(x, y) - (x+y)B(x+1, y)), \end{aligned}$$

where  $x = 2\eta+2k-1$  and  $y = 2\ell-2k$ , which is zero except when  $k = 0 = \ell$ . This verifies the claim.

The right hand side of the equation (3.7) is exactly the expression given in the statement of the Theorem, so its positivity is the condition we were seeking.  $\square$

When  $Y$  is given, we may ask what are the values of  $\eta$  for which the positivity condition of the Theorem holds. It obviously holds when  $\eta$  is large. We can also see that there exists a number  $\eta_Y > 0$  such that it holds if and only if  $\eta > \eta_Y$ . For this we only have to see that if  $E^{(\eta, Y)}$  has a reproducing kernel for some  $\eta > 0$ , then so does  $E^{(\eta+\varepsilon, Y)}$  for all  $\varepsilon > 0$ . Now  $E^{(\eta+\varepsilon, Y)} = L_\varepsilon \otimes E^{(\eta, Y)}$  which shows that the product  $K(z, w) = (1-z\bar{w})^{-2\varepsilon} K_I^{(\eta, Y)}(z, w)$  is a reproducing kernel for  $E^{(\eta+\varepsilon, Y)}$ , and  $K(0, 0) = I$  still holds.

When  $m = 1$ , the condition of the Theorem 3.3 reduces to

$$I - \frac{1}{\eta} Y_1 Y_1^* > 0.$$

In this case, we have  $\eta_Y = \frac{1}{2} \|Y_1 Y_1^*\|$  in terms of the usual matrix norm.

#### 4. Classification of the homogeneous operators in the Cowen-Douglas class

The following theorem together with Theorems 3.1 and 3.2, and Corollary 2.1 gives a complete classification of homogeneous operators in the Cowen-Douglas class.

**THEOREM 4.1.** *All the homogeneous holomorphic Hermitian vector bundles with a reproducing kernel correspond to homogeneous operators in the Cowen-Douglas class. The irreducible ones are the adjoint of the multiplication operator  $M$  on the space  $\mathcal{H}_I^{(\eta,Y)}$  for some  $\eta > 0$  and irreducible  $Y$ . The block matrix  $Y$  is determined up to conjugacy by block diagonal unitaries.*

*Proof.* First we note that by Theorems 3.2 and 3.3 every homogeneous holomorphic Hermitian vector bundle can be written in the form  $(E^{\eta,Y}, I)$  with  $\eta > 0$ . The Hilbert space  $\mathcal{H}_I^{\eta,Y}$  is a subspace of the (trivialized) holomorphic sections of  $(E^{\eta,Y}, I)$  which is the image under the map  $\Gamma_N^{(\eta,Y)}$  of  $\mathbf{A}^{(\eta)}$ . We have to show only that the operator  $M^*$  on  $\mathcal{H}_I^{(\eta,Y)}$  belongs to the Cowen-Douglas class. For this we compute the matrix of  $M$  in an appropriate orthonormal basis.

Each of the Hilbert spaces  $\mathbb{A}^{(\eta+j)}$  ( $0 \leq j \leq m$ ) has a natural orthonormal basis

$$\left\{ e_j^n(z) := \sqrt{\frac{(2\eta+2j)_n}{n!}} z^n : n \geq 0 \right\}.$$

Hence  $\mathbb{A}^{(\eta+j)} \otimes \mathbb{C}^{d_j}$  has the basis  $e_j^n \varepsilon_q^{(j)}$ , where  $\{\varepsilon_q^{(j)} : 1 \leq q \leq d_j\}$  is the natural basis of  $\mathbb{C}^{d_j}$ . The Hilbert space  $\mathbf{A}^{(\eta)}$  then has the orthonormal basis  $e_j^n \varepsilon_{jq}$  with  $\varepsilon_{jq} := \varepsilon_j \otimes \varepsilon_q^{(j)}$ , where  $\{\varepsilon_j : 0 \leq j \leq m\}$  is the natural basis for  $\mathbb{C}^{m+1}$ . Each  $e_j^n \varepsilon_{jq}$  is a function on  $\mathbb{D}$  taking values in  $\mathbb{C}^d$ ; its part in  $\mathbb{C}^{d_j}$  is  $\varepsilon_j \otimes \varepsilon_q^{(j)}$ , and its part in every other  $\mathbb{C}^{d_k}$  ( $k \neq j$ ) is 0. Defining

$$\mathbf{e}_{jq}^n := \Gamma^{(\eta,Y)}(e_j^n \varepsilon_{jq}), \quad (4.1)$$

we have an orthonormal basis for  $\mathcal{H}^{(\eta,Y)}$ .

We identify the “ $K$ -types” in  $\mathcal{H}^{(\eta,Y)}$ , that is, the subspaces on which the representation  $U$  restricted to  $\tilde{\mathbb{K}}$  acts by scalars. For  $k_\theta \in \tilde{\mathbb{K}}$  given by  $k_\theta(z) = e^{i\theta}z$ , we have  $D_{k_\theta}^{(\eta+j)} e_j^n = e^{-i\theta(\eta+j+n)} e_j^n$  on  $\mathbb{A}^{(\eta+j)}$ . By the intertwining property of  $\Gamma^{(\eta,Y)}$ , the basis elements of  $\mathcal{H}^{(\eta,Y)}$  then satisfy  $U_{k_\theta} \mathbf{e}_{jq}^n = e^{-i\theta(\eta+j+n)} \mathbf{e}_{jq}^n$ . It follows that the subspace

$$\mathcal{H}^{(\eta,Y)}(n) := \{f \in \mathcal{H}^{(\eta,Y)} : U_{k_\theta} f = e^{-i\theta(\eta+n)} f\}$$

is spanned by the basis elements  $\{\mathbf{e}_{jq}^{n-j} : 1 \leq q \leq d_j, 0 \leq j \leq \min(m, n)\}$  and  $\mathcal{H}^{(\eta,Y)}$  equals the direct sum  $\oplus_{n \geq 0} \mathcal{H}^{(\eta,Y)}(n)$ .

Clearly, the operator  $M$  maps each  $\mathcal{H}^{(\eta,Y)}(n)$  to  $\mathcal{H}^{(\eta,Y)}(n+1)$ . We write  $M(n)$  for the matrix of the restriction of  $M$  to  $\mathcal{H}^{(\eta,Y)}(n)$ , that is,

$$M \mathbf{e}_{jq}^{n-j} = \sum_{\ell, r} M(n)_{(\ell r)(jq)} \mathbf{e}_{\ell r}^{n+1-\ell}. \quad (4.2)$$

We write  $e_{(jq), (st)}^{n-j}(z)$  for the  $(s, t)$ -component ( $0 \leq s \leq \min(m, n)$ ,  $1 \leq t \leq d_s$ ) of the function  $\mathbf{e}_{jq}^{n-j}$  taking values in  $\mathbb{C}^d$ . This can be regarded as a matrix of monomials in  $z$ . The coefficients of these monomials form a numerical matrix which we denote by  $E(n)$ .

Applying the operator  $M$ , which is multiplication by  $z$ , to the monomials does not change their coefficients. Therefore, equation (4.2) amounts to the matrix equality

$$E(n) = E(n+1)M(n). \quad (4.3)$$

We use (4.1) to compute  $E(n)$  explicitly. The part in  $\mathbb{C}^{d_j}$  of the vector valued function  $e_j^{n-j} \varepsilon_{jq}$  is  $e_j^{n-j} \varepsilon_q^{(j)}$ , its part in  $\mathbb{C}^{d_k}$  with  $k \neq j$  is 0. So (3.2) gives, for the part of  $\mathbf{e}_{jq}^{n-j}$  ( $0 \leq j \leq m$ ) in  $\mathbb{C}^{d_\ell}$ ,

$$(\mathbf{e}_{jq}^{n-j}(z))_\ell = \begin{cases} c(\eta, \ell, j, n) z^{n-\ell} (Y_\ell \cdots Y_{j+1}) \varepsilon_q^{(j)} z^{n-\ell} & \text{if } \ell \geq j \\ 0 & \text{if } \ell < j, \end{cases} \quad (4.4)$$

where the constant  $c(\eta, \ell, j, n)$  is the coefficient of  $z^{n-\ell}$  in

$$\frac{1}{(\ell-j)!} \frac{1}{(2\eta+2j)_{\ell-j}} \left(\frac{d}{dz}\right)^{\ell-j} e_j^{n-j}(z) = \frac{1}{(\ell-j)!} \frac{1}{(2\eta+2j)_{\ell-j}} \sqrt{\frac{(2\eta+2j)_n}{n!}} \left(\frac{d}{dz}\right)^{\ell-j} z^{n-j}.$$

We can regard  $E(n)$  as a block matrix with blocks  $E(n)_{j\ell}$  of size  $d_j \times d_\ell$ . The  $(q, r)$  entry of  $E(n)_{j\ell}$  being  $E(n)_{(jq)(\ell r)}$  defined above. Then equation (4.4) says that

$$E(n)_{j\ell} = \begin{cases} c(\eta, \ell, j, n) Y_\ell \cdots Y_{j+1} & \text{if } \ell \geq j \\ 0 & \text{if } \ell < j. \end{cases}$$

Now, we consider the behavior of  $c(\eta, \ell, j, n)$  for large  $n$ . First, since

$$\sqrt{\frac{(2\eta+2j)_{n-j}}{(n-j)!}} \left(\frac{d}{dz}\right)^{\ell-j} z^{n-j} = \frac{\sqrt{(n-j)!(2\eta+2j)_{n-j}}}{(n-\ell)!} z^{n-\ell},$$

it follows that

$$c(\eta, \ell, j, n) = \frac{1}{(2\eta+2j)_{\ell-j}(\ell-j)!} \frac{\sqrt{\Gamma(n-j+1)\Gamma(2\eta+j+n)}}{\sqrt{\Gamma(2\eta+2j)\Gamma(n-\ell+1)}}.$$

From Stirling's formula, we obtain

$$\begin{aligned} c(\eta, \ell, j, n) &\sim \frac{1}{\sqrt{\Gamma(2\eta+2j)(2\eta+2j)_{\ell-j}(\ell-j)!}} \frac{\sqrt{(e^{-n}n^{n-j+\frac{1}{2}})(e^{-n}n^{n+2\eta+j-\frac{1}{2}})}}{e^{-n}n^{n-\ell+\frac{1}{2}}} \\ &\sim \frac{\sqrt{\Gamma(2\eta+2j)}}{\Gamma((\ell-j+1)\Gamma(2\eta+2j+\ell))} n^{\eta-\frac{1}{2}+\ell}. \end{aligned}$$

If we define the block matrix  $E$  by

$$E_{\ell j} = \begin{cases} \frac{\sqrt{\Gamma(2\eta+2j)}}{\Gamma((\ell-j+1)\Gamma(2\eta+2j+\ell))} Y_\ell \cdots Y_{j+1} & \text{if } \ell \geq j \\ 0 & \text{if } \ell < j \end{cases}$$

and the diagonal block matrix  $D(n)$  by  $D(n)_{\ell\ell} = n^\ell I_{d_\ell}$  then we can write our result as

$$E(n) \sim n^{\eta-\frac{1}{2}} D(n) E$$

From (4.3), for large  $n$ , it follows that

$$\begin{aligned} M(n) &= E(n+1)^{-1} E(n) \\ &\sim \left(\frac{n}{n+1}\right)^{\eta-\frac{1}{2}} E^{-1} D(n+1)^{-1} D(n) E \\ &\sim I + O(1/n). \end{aligned} \quad (4.5)$$

Therefore, the operator  $M$  which is a “weighted block shift” is the direct sum of an ordinary (unweighted) block shift and a Hilbert - Schmidt operator. Hence  $M$  is bounded and standard results from Fredholm theory ensure that the adjoint operator  $M^*$  is in the Cowen-Douglas class  $B_d(\mathbb{D})$ .  $\square$

The similarity classes of the homogeneous Cowen-Douglas operators are easily described in terms of the parameter  $\eta$  and the multiplicities  $d_0, \dots, d_m$ . For a somewhat smaller class of operators, the similarity classes were described in [11].

**THEOREM 4.2.** *The multiplication operator  $M$  on  $\mathcal{H}_I^{(\eta,Y)}$  and on  $\mathcal{H}_I^{(\eta',Y')}$  are similar if and only if the blocks in  $Y$  and  $Y'$  are of the same size and  $\eta = \eta'$ .*

*Proof.* To prove the forward direction, first we show that  $\Gamma^{(\eta,Y)}$  maps  $\mathbf{A}^{(\eta)}$  onto itself, that is,  $\mathbf{A}^{(\eta)} = \mathcal{H}_I^{(\eta,Y)}$  as linear spaces. The derivative  $\frac{d}{dz} : \mathbb{A}^{(\alpha)} \rightarrow \mathbb{A}^{(\alpha+1)}$  defines a surjective bounded linear operator for any  $\alpha > 0$ . For any  $f \in \mathbf{A}^{(\eta)}$ ,

$$(\Gamma^{(\eta,Y)} f)_\ell = \sum_{j=0}^{\ell} (\Gamma^{(\eta,Y)} f_j)_\ell$$

and (3.2) shows that each term of the sum is in  $d_\ell \mathbb{A}^{(\eta+\ell)}$ . On the other hand, given  $g = (g_1, \dots, g_m) \in \mathbf{A}^{(\eta)}$ , we find  $f \in \mathbf{A}^{(\eta)}$  satisfying  $\Gamma^{(\eta,Y)} f = g$ . The functions  $f_0, \dots, f_d$  are determined recursively. Suppose, we have already determined  $f_j$ ,  $j < \ell$ . Then from the definition of the map  $\Gamma^{(\eta,Y)}$ , we see that taking

$$f_\ell = g_\ell - \sum_{j=0}^{\ell-1} (\Gamma^{(\eta,Y)} f_j)_\ell$$

we have the required  $f$ . Clearly,  $M : \mathbf{A}^{(\eta)} \rightarrow \mathbf{A}^{(\eta)}$  is similar to  $M : H_I^{(\eta,Y)} \rightarrow \mathcal{H}_I^{(\eta,Y)}$  via the map  $f \mapsto f$ , which is bounded and invertible by the Closed graph theorem.

For the proof in the other direction, let  $K(n) \subseteq \mathbf{A}^{(\eta)} = \oplus_{j=0}^m d_j \mathbb{A}^{(\eta+j)}$  be the linear span of the vectors  $\{e_{jq}^n : 0 \leq j \leq m, 1 \leq q \leq d_j\}$ . The multiplication operator  $M$  on  $\mathbf{A}^{(\eta)}$  maps  $K(n)$  into  $K(n+1)$ . If  $M_n$  is the matrix representing  $M|_{K(n)} : K(n) \rightarrow K(n+1)$  then  $M$  is a block shift with blocks  $\{M_n : n \geq 0\}$ , which are diagonal matrices of size  $d \times d$ . Let  $M'$  be the multiplication operator on  $\mathbf{A}^{(\eta')} = \oplus_{j=0}^{m'} d'_j \mathbb{A}^{(\eta'+j)}$  with a similar block decomposition. Assume without loss of generality that  $\eta' > \eta$ . Suppose  $L : \mathbf{A}^{(\eta)} \rightarrow \mathbf{A}^{(\eta')}$  is a bounded and invertible linear map consisting of  $d \times d$  blocks with  $LM = M'L$ . Then  $d = d_0 + \dots + d_m = \text{codim}(\text{ran } M) = \text{codim}(\text{ran } M') = d'_0 + \dots + d'_{m'}$ .

It then follows that  $L_{0k} = 0$  for all  $k \geq 1$  and consequently  $L_{00}$  is non-singular. We also have  $L_{nn}M_{n-1} = M'_{n-1}L_{n-1n-1}$  from which it follows that

$$L_{nn} = M'_{n-1} \cdots M'_0 L_{00} M_0^{-1} \cdots M_n^{-1} = F'_n L_{00} F_n^{-1},$$

where  $F_n = M_0 \cdots M_{n-1}$  and  $F' = M'_{n-1} \cdots M'_0$  are diagonal matrices. The diagonal elements are

$$F(n)_{kk} = \sqrt{\frac{(2\eta + 2j(k))_n}{n!}} \quad \left( \text{respectively, } F'(n)_{\ell\ell} = \sqrt{\frac{(2\eta' + 2j'(\ell))_n}{n!}} \right),$$

where  $j(k) = j$  if  $d_0 + \dots + d_{j-1} < k \leq d_0 + \dots + d_j$ . By Stirling's formula, we have

$$(L_{nn})_{\ell k} = (F'_n)_{\ell\ell} (L_{00})_{\ell k} (F_n^{-1})_{kk} \sim n^{\eta' - \eta + j'(\ell) - j(k)} (L_{00})_{\ell k}.$$

Since  $L_{00}$  is nonsingular, for any  $k$  with  $j(k) = 0$ , there is an  $\ell$  such that  $(L_{00})_{\ell k} \neq 0$ . Now, unless  $\eta = \eta'$ , we have  $(L_{nn})_{\ell k} \rightarrow \infty$  contradicting the boundeness of  $L$ . Therefore, we have  $\eta = \eta'$  and  $(L_{nn})_{\ell k} \sim n^{j'(\ell) - j(k)} (L_{00})_{\ell k}$ . Take all those  $k$  for which  $j(k) = 0$ . For each of these, we can find a different  $\ell_k$  such that  $(L_{00})_{\ell_k k} \neq 0$ . (The columns of the nonsingular matrix  $L_{00}$  with these indices are linearly independent and therefore cannot have only zeros in more than  $d - k$  slots.) Again, unless  $j'(\ell_k) = 0$ , we have  $(L_{nn})_{\ell_k k} \rightarrow \infty$ . This shows that  $d'_0 \geq d_0$ . Similarly,  $d'_j \geq d_j$ ,  $1 \leq j \leq m$ . From the equality  $\sum_{j=0}^{m'} d'_j = \sum_{j=0}^m d_j$ , it follows that  $m' = m$  and  $d'_j = d_j$  for  $j = 1, \dots, m$ .  $\square$

The following Corollary, the proof of which is evident, implies that polynomially bounded homogeneous operators in the Cowen-Douglas class are similar to contractions.

**COROLLARY 4.1.** *A homogeneous operator in the Cowen-Douglas class is either similar to a contraction or it is not power bounded.*

## 5. EXAMPLES

In this last Section, we discuss how some formerly known examples fit into the present framework.

**5.1. The case of  $d_0 = d_1 = \dots = d_m = 1$ .** This case was already studied in [7]. Here each  $Y_j$  is a number, non-zero in the irreducible case. The unitaries implementing the equivalence are diagonal, and clearly the conjugacy class  $[Y]$  under these has exactly one representative with  $y_j > 0$ ,  $1 \leq j \leq m$ . The positive  $m + 1$  - tuples satisfying the condition given in Theorem 3.3 give a parametrization of homogeneous Cowen-Douglas operators. For each one,  $K(0, 0) = I$  and  $J_g^{(\eta, Y)}$  is given by the formula (2.4).

Another good parametrization is possible with the aid of Theorem 3.1. All possible  $Y$ -s are now conjugate under diagonal unitaries  $A$ , so we may fix an arbitrary  $Y^{(0)}$  (for example,  $y_j = 1$  for all  $j$ , or, as in [7],  $y_j = j$  for all  $j$ ). Take any positive diagonal matrix  $N$  with diagonal elements  $1 = \mu_0, \mu_1, \dots, \mu_m$ . By Proposition 2.2,  $Y^{(0)}, N$  and  $Y^{(0)}, N'$  give isomorphic vector bundles if and only if  $A$  is diagonal and hence  $N = N'$ . It follows that the positive numbers  $\eta, \mu_1, \dots, \mu_m$  give a parametrization of the homogeneous operators in the Cowen-Douglas class  $B_{m+1}(\mathbb{D})$ . Here  $J_g^{(\eta, Y^{(0)})}$  depends only on  $\eta$  and  $K_N^{(\eta, Y^{(0)})}(0, 0)$  is given by the formula in Theorem 3.1. This is the parametrization used in [7].

In the case  $m = 1$ , for any  $d_0$  and  $d_1$ , the class  $[Y]$  always contains a member for which  $Y$  is diagonal. So, the corresponding bundle is reducible unless  $d_0 = d_1 = 1$ . When  $m = 2$ , it is easy to see that  $d_0 = 2$  or  $d_2 = 2$  gives only reducible bundles. So, the first non-trivial case occurs (apart from the case  $d_0 = d_1 = d_2 = 1$ , which has been dealt with previously) when  $d_0 = d_2 = 1$ ,  $d_1 = 2$ .

**5.2. The case of  $(d_0, d_1, d_2) = (1, 2, 1)$ .** For this case, again there are two natural parametrizations. Conjugating  $Y$  with a block-diagonal unitary having blocks  $u_0, U_1, u_2$  changes  $Y_1, Y_2$  into  $U_1 Y_1 u_0^{-1}$ ,  $u_2 Y_2 U_1^{-1}$ . Now,  $U_1$  can be chosen so that  $Y_1 = \begin{pmatrix} a \\ 0 \end{pmatrix}$ . Then  $u_0, u_2$  and a scalar factor in front of  $U_1$  can be found with  $a \geq 0$  and  $Y = \begin{pmatrix} b & c \end{pmatrix}$  with  $b, c \geq 0$ . We have irreducibility if and only if  $a, b, c \neq 0$  and no two such triples give equivalent  $Y$ -s. So, we have a parametrization of the irreducible  $E^{(\eta, Y)}$  by four arbitrary non-zero parameters. There is a reproducing kernel (and hence an operator in  $B_4(\mathbb{D})$ ) if and only if the the right hand side of the equation (3.7) is positive; in terms of the parameters, this is

$$\begin{aligned} a^2 &< 2\eta \\ b^2 &< \frac{2\eta + 2}{1 - \frac{a^2}{2(2\eta + 1)}} \\ c^2 &< 2\eta + 2 \end{aligned}$$

The positive quadruple  $(\eta, a, b, c)$  subject to this condition parametrizes the homogeneous operators in  $B_4(\mathbb{D})$ . In each case,  $K(0, 0) = I$  and  $J_g$  can be expressed in terms of the parameters using (2.4).

The other parametrization of the  $(d_0, d_1, d_2) = (1, 2, 1)$  case is found using Theorem 3.1. Simple arguments show that  $Y$  can always be conjugated by a block diagonal  $A$  so that  $Y_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $Y_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  or  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . When  $Y_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ , the bundle will be reducible for any choice of Hermitian structure. So, we can fix  $Y^{(0)}$  with  $Y_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $Y_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ . The block diagonal  $A$  that conjugates this  $Y^{(0)}$  to itself is a diagonal matrix with  $(p, p, q, p)$  on the diagonal. If  $N$  is any positive diagonal  $\text{diag}(n_0, N_1, n_2)$  with  $n_0 = 1$ ,  $N_1 = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \gamma \end{pmatrix}$  and  $n_2 \geq 0$ , then we can ensure  $n_1 = 1 = n_2$  and  $\alpha, \beta, \gamma > 0$  after conjugating by an  $A$ . Thus the homogeneous bundles with reproducing kernel (and hence the homogeneous operators in  $B_4(\mathbb{D})$ ) of type  $(1, 2, 1)$  are now parametrized by four positive numbers  $(\eta, \alpha, \beta, \gamma)$  subject to the condition  $\beta^2 < \alpha\gamma$ .

By a different construction, a large subset of these examples already occurs in [12].

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